NON-FACTORISATION OF ARF-KERVAIRE CLASSES
THROUGH $\mathbb{R}P^{\infty} \wedge \mathbb{R}P^{\infty}$

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Abstract. As an application of the upper triangular technology method
of [8] it is shown that there do not exist stable homotopy classes of $\mathbb{R}P^{\infty} \wedge \mathbb{R}P^{\infty}$ in
dimension $2s+1 - 2$ with $s \geq 2$ whose composition with the Hopf
map to $\mathbb{R}P^{\infty}$ followed by the Kahn-Priddy map gives an element in the
stable homotopy of spheres of Arf-Kervaire invariant one.

1. Introduction

1.1. For $n > 0$ let $\pi_n(\Sigma^\infty S^0)$ denote the $n$-th stable homotopy group of $S^0$, the
0-dimensional sphere. Via the Pontrjagin-Thom construction an element of
this group corresponds to a framed bordism class of an $n$-dimensional framed
manifold. The Arf-Kervaire invariant problem concerns whether or not there
exists such a framed manifold possessing a Kervaire surgery invariant which
is non-zero (modulo 2). In [4] it is shown that this can happen only when
$n = 2s+1 - 2$ for some $s \geq 1$. Resolving this existence problem is an important
unsolved problems in homotopy theory (see [8] for a historical account of the
problem together with new proofs of all that was known up to 2008). Recently
important progress has made ([5]; see also [2], [3]) which shows that
$n = 126$ is the only remaining possibility for existence (more details may be found in
the survey article [9].

In view of the renewed interest in the Arf-Kervaire invariant problem it
may be of interest to describe a related non-existence result. An equivalence
formulation (see [8] § 1.8) is that there exists a stable homotopy class $\Theta : \Sigma^\infty S^{2s+1-2} \to \Sigma^\infty \mathbb{R}P^{\infty}$ with mapping cone $\text{Cone}(\Theta)$ such that the Steenrod
operation

$$Sq^2 : H^{2s-1}(\text{Cone}(\Theta); \mathbb{Z}/2) \cong \mathbb{Z}/2 \to H^{2s+1-1}(\text{Cone}(\Theta); \mathbb{Z}/2)$$

is non-trivial. Using the upper triangular technology (UTT) of [8] we shall
prove the following result:

Theorem 1.2.

Let $H : \Sigma^\infty \mathbb{R}P^{\infty} \wedge \mathbb{R}P^{\infty} \to \Sigma^\infty \mathbb{R}P^{\infty}$ denote the map obtained by applying
the Hopf construction to the multiplication on $\mathbb{R}P^{\infty}$. Then, if $s \geq 2$, there
does not exist a stable homotopy class

$$\tilde{\Theta} : \Sigma^\infty S^{2s+1-2} \to \Sigma^\infty \mathbb{R}P^{\infty} \wedge \mathbb{R}P^{\infty}$$

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such that the composition $\Theta = H \cdot \tilde{\Theta}$ is detected by a non-trivial $Sq^2$ as in §1.1.

In §2.2 this result will be derived as a simple consequence of the UTT relations ([8] Chapter Eight). The basics of the UTT method are sketched in §2.1. Doubtless there are other ways to prove Theorem 1.2 (for example, from the results of [10]; see also [8] Chapter Two) but it provides an elegant application of UTT.

2. Upper triangular technology (UTT)

2.1. Let $F_{2n}(\Omega^2 S^3)$ denote the $2n$-th filtration of the combinatorial model for $\Omega^2 S^3 \simeq W \times S^1$. Let $F_{2n}(W)$ denote the induced filtration on $W$ and let $B(n)$ be the Thom spectrum of the canonical bundle induced by $f_n : \Omega^2 S^3 \rightarrow BO$, where $B(0) = S^0$ by convention. From [7] one has a 2-local, left $bu$-module homotopy equivalence of the form

$$
\bigvee_{n \geq 0} bu \land \Sigma^{4n} B(n) \xrightarrow{\sim} bu \land bo.
$$

Therefore, if $\Theta$ is as in §1.1, then

$$(bu \land bo)_\ast (\text{Cone}(\Theta)) \cong \bigoplus_{n \geq 0} (bu_\ast (\text{Cone}(\Theta) \land \Sigma^{4n} B(n))).$$

Let $\alpha(k)$ denote the number of 1’s in the dyadic expansion of the positive integer $k$. For $1 \leq k \leq 2^s - 1$ and $2^s \geq 4k - \alpha(k) + 1$ there are isomorphisms of the form ([8] Chapter Eight §4)

$$bu_{2s+1-1}(C(\Theta) \land \Sigma^{4k} B(k)) \cong bu_{2s+1-1}(\mathbb{R}P^\infty \land \Sigma^{4k} B(k)) \cong V_k \oplus \mathbb{Z}/2^{s-4k+\alpha(k)}$$

where $V_k$ is a finite-dimensional $\mathbb{F}_2$-vector space consisting of elements which are detected in mod 2 cohomology (i.e. in filtration zero, represented on the $s = 0$ line) in the mod 2 Adams spectral sequence. The map $1 \land \psi^3 \land 1$ on $bu \land bo \land C(\Theta)$ acts on the direct sum decomposition like the upper triangular matrix

\[
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & \cdots \\
0 & 9 & 1 & 0 & 0 & \cdots \\
0 & 0 & 9^2 & 1 & 0 & \cdots \\
0 & 0 & 0 & 9^3 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

\footnote{In [8] and related papers I consistently forgot what I had written in my 1998 McMaster University notes “On $bu_\ast (BD_8)$”. Namely, in the description of Mahowald’s result I stated that $\Sigma^{4n} B(n)$ was equal to the decomposition factor $F_{4n}/F_{4n-1}$ in the Snaith splitting of $\Omega^2 S^3$. Although this is rather embarrassing, I got the homology correct so that the results remain correct upon replacing $F_{4n}/F_{4n-1}$ by $\Sigma^{4n} B(n)$ throughout! I have seen errors like this in the World Snooker Championship where the no.1 player misses an easy pot by concentrating on positioning the cue-ball. In mathematics such errors are inexcusable whereas in snooker they only cost one the World Championship.}
In other words \((1∧ψ^3∧1)_*\) sends the \(k\)-th summand to itself by multiplication by \(9^{k-1}\) and sends the \((k-1)\)-th summand to the \((k-2)\)-th by a map

\[
(t_{k,k-1})_* : V_k \oplus \mathbb{Z}/2^{s-4k+\alpha(k)} \longrightarrow V_{k-1} \oplus \mathbb{Z}/2^{s-4k+\alpha(k-1)}
\]

for \(2 \leq k \leq 2^{s-1} - 1\) and \(2^s \geq 4k - \alpha(k) + 1\). The right-hand component of this map is injective on the summand \(\mathbb{Z}/2^{s-4k+\alpha(k)}\) and annihilates \(V_k\).

It is shown in [6] (also proved by UTT in ([8] Chapter Eight when \(s \geq 2\)) that \(Θ\) corresponds to a stable homotopy class of Arf-kervaire invariant one if and only if it is detected by the Adams operation \(ψ^3\) on \(τ \in bu_{2s+1-1}(\text{Cone}(Θ))\), an element of infinite order.

From these properties and the formula for \(ψ^3(τ)\) one easily obtains a series of equations ([8] §8.4.3) for the components of \((η ∧ 1 ∧ 1)_*(τ)\) where \(η : S^0 \longrightarrow bu\) is the unit of \(bu\)-spectrum. Here we have used the isomorphism \(bu_{2s+1-1}(C(Θ)) \cong bo_{2s+1-1}(C(Θ))\) since, strictly speaking, the latter group is the domain of \((η ∧ 1 ∧ 1)_*\). It is shown in ([8] Theorem 8.4.7) that this series of equations implies that the \(bu_{2s+1-1}(\text{Cone}(Θ) ∧ Σ^2 B(2^{s-2}))\)-component of \((η ∧ 1 ∧ 1)_*(τ)\) is non-trivial and gives some information on the identity of this non-trivial element.

It is this information which we shall now use to prove Theorem 1.2.

### 2.2. Proof of Theorem 1.2

Suppose, for a contradiction, that \(Θ\) and \(Θ\) exist. We must assume that \(s \geq 2\) because the UTT results of ([8] Theorem 8.4.7) are only claimed for this range.

The mod 2 cohomology of \(Σ^2 B(2^{s-2})\) is given by the \(\mathbb{F}_2\)-vector space with basis \(\{z_{2^j+2j}, 0 \leq j \leq 2^{s-1} - 2; z_{2^j+3+2k}, 0 \leq k \leq 2^{s-1} - 2\}\) on which the left action by \(Sq^1\) and \(Sq^{0,1} = Sq^1 Sq^2 + Sq^2 Sq^1\) are given by \(Sq^1(z_{2^j+2j}) = z_{2^{j+1}+3+2(j-1)}\) for \(1 \leq j \leq 2^{s-1} - 1\) and \(Sq^{0,1}(z_{2^j+2j}) = z_{2^{j+1}+3+2j}\) for \(0 \leq j \leq 2^{s-1} - 2\) and \(Sq^1, Sq^{0,1}\) are zero otherwise. This cohomology module is the \(\mathbb{F}_2\)-dual of the “lightning flash” module depicted in ([1] p.341).

Now consider the two 2-local Adams spectral sequences

\[
E_2^{s,t} = Ext_B^{s,t}(H^*(C(Θ); \mathbb{Z}/2) \otimes H^*(Σ^2 B(2^{s-2}; \mathbb{Z}/2), \mathbb{Z}/2)
\]

\[
\Rightarrow bu_{t-s}(C(Θ) ∧ Σ^2 B(2^{s-2}))
\]

which collapses and

\[
\tilde{E}_2^{s,t} = Ext_B^{s,t}(H^*(C(Θ); \mathbb{Z}/2) \otimes H^*(Σ^2 B(2^{s-2}; \mathbb{Z}/2), \mathbb{Z}/2)
\]

\[
\Rightarrow bu_{t-s}(C(Θ) ∧ Σ^2 B(2^{s-2}))
\]

where \(B\) is the exterior subalgebra of the mod 2 Steenrod algebra generated by \(Sq^1\) and \(Sq^{0,1}\).

To fit in with the notation of ([8] Theorem 8.4.7) set \(s = q + 2\) in Theorem 1.2. As mentioned in §2.1, it is shown in ([8] Theorem 8.4.7) that the
component of $(\eta \land 1 \land 1)_\ast(\iota)$ lying in

$$bu_{2^t+3-1}(C(\Theta) \land \Sigma^2 B(2^{s-2}))$$

\[ \cong bu_{2^t+3-1}(\mathbb{R} P^\infty \land \Sigma^2 B(2^{s-2})) \]

\[ \cong \text{Ext}_B^{0,2^t+3-1}(H^*(\mathbb{R} P^\infty; \mathbb{Z}/2) \otimes H^*(\Sigma^2 B(2^{s-2}; \mathbb{Z}/2), \mathbb{Z}/2) \]

\[ \subseteq \text{Hom}(\oplus_{u+v=2^{t+3}-1} H^u(\mathbb{R} P^\infty; \mathbb{Z}/2) \otimes H^v(\Sigma^2 B(2^{s-2}; \mathbb{Z}/2), \mathbb{Z}/2) \]

corresponds to a homomorphism $f$ such that $f(x^{2^{t+2}-1} \otimes z_{2^{t+2}})$ is non-trivial.

The factorisation $\Theta = H \cdot \tilde{\Theta}$ implies that there exists $h \in \tilde{E}_2^{0,2^t+3-1} \subseteq \tilde{E}_2^{0,2^t+3-1}$ such that $H_s(h) = f$. On the other hand

$$\tilde{E}_2^{0,2^t+3-1} \cong \text{Ext}_B^{0,2^t+3-1}(H^*(\mathbb{R} P^\infty \land \mathbb{R} P^\infty; \mathbb{Z}/2) \otimes H^*(\Sigma^2 B(2^{s-2}; \mathbb{Z}/2), \mathbb{Z}/2).$$

Therefore the homomorphism $f$ satisfies $(H \land 1)_\ast (\iota)(x^{2^{t+2}-1} \otimes z_{2^{t+2}}) = f(x^{2^{t+2}-1} \otimes z_{2^{t+2}}) \neq 0$. However

$$\text{Ext}_B^{0,2^t+3-1}(H^*(\mathbb{R} P^\infty; \mathbb{Z}/2) \otimes H^*(F_{2^{t+2}}/F_{2^{t+2}-1}; \mathbb{Z}/2), \mathbb{Z}/2)$$

satisfies $(H \land 1)_\ast (\iota)(x^{2^{t+2}-1} \otimes z_{2^{t+2}}) = h(\sum_{a=1}^{2^{t+2}-2} x^a \otimes x^{2^{t+2}-a-1} \otimes z_{2^{t+2}})$.

On the other hand

$$Sq^1(x^a \otimes x^{2^{t+2}-2-a} \otimes z_{2^{t+2}})$$

$$= \alpha(x^a \otimes x^{2^{t+2}-1-a} \otimes z_{2^{t+2}} + x^{a+1} \otimes x^{2^{t+2}-2-a} \otimes z_{2^{t+2}}) + x^a \otimes x^{2^{t+2}-2-a} \otimes Sq^1(z_{2^{t+2}})$$

$$= \alpha(x^a \otimes x^{2^{t+2}-1-a} \otimes z_{2^{t+2}} + x^{a+1} \otimes x^{2^{t+2}-2-a} \otimes z_{2^{t+2}})$$

since $Sq^1(z_{2^{t+2}})$ is trivial. Therefore

$$f(x^{2^{t+2}-1} \otimes z_{2^{t+2}}) \in h(\text{Im}(Sq^1)) \equiv 0$$

because $h$ is a $B$-module homomorphism and $Sq^1$ is trivial on $\mathbb{Z}/2$. \[ \square \]

**Remark 2.3.** When $s = 2, 3$ in the situation of Theorem 1.2 there is a map $\alpha : \Sigma^\infty \mathbb{R} P^\infty \land \mathbb{R} P^\infty \longrightarrow \Sigma^\infty \mathbb{R} P^\infty$ but it is just not equal to $H!$ In the loopspace structure of $Q\mathbb{R} P^\infty$ form the product minus the two projections to give a map $\mathbb{R} P^\infty \times \mathbb{R} P^\infty \longrightarrow Q\mathbb{R} P^\infty$ which factors through the smash...
product. The adjoint of this factorisation is $\alpha$. Then the smash product of two copies of a map of Hopf invariant one $\Sigma^\infty S^{2s-1} \to \Sigma^\infty \mathbb{R}P^\infty$ composed with $\alpha$ is detected by $Sq^{2s}$ on its mapping cone (see [10]).

REFERENCES