ADMISSIBLE REPRESENTATIONS OF $GL_n K$ AND THEIR MONOMIAL RESOLUTIONS

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1. Introduction

This document is a sketch of a method of constructing all admissible representations of $GL_2 K$ of a local field (representations defined over any algebraically closed field) and all automorphic representations when $K$ is a number field in terms of uniquely defined chain complexes in the homotopy category of monomial complexes. This construction makes sense for all $GL_n K$ but my current knowledge of Bruhat-Tits buildings runs only to Conjecture 4.3. In the later sections I sketch how the definition of epsilon factors and L-functions and Galois descent should go.

I am sending out this essay to a few mathematicians who might be interested in getting themselves, their postdocs or their students involved. I am intending to prepare the 300-page typescript of the details of results so far into a monograph fit for publication. The only drawbacks being that, as a retired professor in the UK, I have neither access to resources nor do I give many hours per days to the development of this project.

2. Monomial resolutions

2.1. Monomial resolutions of admissible representations

Let $k$ be an algebraically closed field with a topology, not necessarily of characteristic zero.

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Let $G$ be an arbitrary locally compact Hausdorff group such that the compact open subgroups form a basis for the neighborhoods of the identity. In particular we are thinking of locally $p$-adic Lie groups such as $GL_n K$ where $K$ is a $p$-adic local field. The subgroups of such a group which we shall concentrate on are the subgroups $H \subseteq G$ which are compact open modulo the centre $Z(G)$. This means that the image of $H$ in $G/Z(G)$, with the induced topology, is compact open. Since we shall be emphasising $GL_n K$ whose centre is the group $K^*$ consisting of the scalar matrices a good example of a compact open modulo the centre subgroup is $K^* \cdot GL_n \mathcal{O}_K \subseteq GL_n K$ where $\mathcal{O}_K$ denotes the valuation ring of $K$.

If $H \subseteq G$ is a compact modulo the centre subgroup then we write $\hat{H}$ for the multiplicative group of continuous group homomorphisms $\hat{H} = \text{Hom}_{cts}(H, k^*)$. When $k = \mathbb{C}$ then $\mathbb{C}^*$ has the discrete topology. The example $K^*$ is easy to describe since there is a topological isomorphism

$$K^* \cong \mathcal{O}_K^* \times \mathbb{Z} \langle \pi_K \rangle$$

where $\pi_K$ is a uniformiser of $K$. A continuous homomorphism to $\mathbb{C}^*$ in this case means a homomorphism which is of finite order when restricted to the group of units $\mathcal{O}_K^*$ and on the infinite cyclic group generated by $\pi_K$ it is given by $\pi_K^n \mapsto x^n$ for some $x \in \mathbb{C}^*$.

Let $\phi : Z(G) \longrightarrow k^*$ be a fixed choice of continuous central character.

Let $\mathcal{M}_{G, \phi}$ denote the set of pairs $(J, \phi)$ with $J \subseteq G$ a compact modulo the centre subgroup containing $Z(G)$ and $\phi \in \hat{J}$ such that $\text{Res}^J_{Z(G)}(\phi) = \phi$. The set $\mathcal{M}_{G, \phi}$ is endowed with the usual partial order in which $(J, \phi) \leq (J', \phi')$ if and only if $J \subseteq J'$ and $\text{Res}^J_{Z(G)}(\phi') = \phi$. Also $G$ acts on $\mathcal{M}_{G, \phi}$ (on the left) by the formula $g(J, \phi) = (gJg^{-1}, (g^{-1})^*(\phi))$ where $(g^{-1})^*(\phi)(gJg^{-1}) = \phi(j)$. The $G$-stabiliser of $(J, \phi)$ under this action is denoted by $N_G(J, \phi)$ and the $G$-orbit of $(J, \phi)$ is denoted by $(J, \phi)^G$. Associated to $(J, \phi)$ is the $k[J]$-module $k_\phi$ given by $k$ on which $J$ acts via $j(z) = \phi(j) \cdot z$ for all $z \in k$.

An $G$-Line Bundle$^1$ is a continuous $k[G]$-module $M$ together with a decomposition into the direct sum of one-dimensional subspaces

$$M = \bigoplus_{a \in A} M_a$$

where the $M_a$’s are permuted by the $G$-action and the stabiliser of each $M_a$ is a pair $(H_a, \phi_a) \in \mathcal{M}_{G, \phi}$. In particular each $H_a$ is compact modulo the centre. The subspaces $M_a$’s are called the Lines of $M$. The pair $(H_a, \phi_a)$ is called the stabilising pair of $M_a$.

For a pair $(H, \phi) \in \mathcal{M}_{G, \phi}$ the $G$-Line Bundle denoted by $\text{Ind}^G_H(\phi)$ is given by the direct sum of lines $L_g$ for $g \in G/H$ where $L_g$ is $k$ upon which $gHg^{-1}$ acts by the formula $gHg^{-1} \cdot z = \phi(h)z$ and $g \in G$ sends $L_g$ to $L_{gg'}$ by the identity map on $k$. As in the case of linear representations this construction

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$^1$The capital letters are chosen there to distinguish the Line Bundle from the familiar vector bundle terminology.
with be called the G-Line Bundle induced from \((H, \phi)\). The stabilising pair of \(L_g\) is \(g(H, \phi) = (gHg^{-1}, (g^{-1})^*(\phi))\). Since \((H, \phi) \in \mathcal{M}_{G, \phi}\) the Line Bundle \(\text{Ind}^{G}_{H}(\phi)\) may be topologised to be continuously isomorphic to the compact induction \(c - \text{Ind}^{G}_{H}(\phi)\) ([1] p.19).

For each \((J, \phi) \in \mathcal{M}_{G, \phi}\) set

\[
M^{((J, \phi))} = \bigoplus_{\alpha \in A, (J, \phi) \leq (H_\alpha, \phi_\alpha)} M_{\alpha},
\]

which is a subspace of \(M\) called the \((J, \phi)\)-fixed points of \(M\). A morphism \(f : M = \bigoplus_{\alpha \in A} M_{\alpha} \longrightarrow \bigoplus_{\beta \in B} M'_{\beta} = M'\) between two G-Line Bundles is a continuous \(k[G]\)-module homomorphism such that

\[
f(M^{((J, \phi))}) \subseteq (M')^{((J, \phi))}
\]

for all \((J, \phi) \in \mathcal{M}_{G, \phi}\). This defines a category \(k[G, \phi]_{\text{mon}}\) and the union of categories over all \(\phi\) is just denoted by \(k[G]_{\text{mon}}\). These are additive categories; this means, for example, that \(\text{Hom}_{k[G, \phi]_{\text{mon}}}(M, M')\) is an \(k\)-vector space.

A monomial complex is a chain complex of continuous \(k[G]\)-modules

\[
\cdots \longrightarrow C_n \xrightarrow{d} C_{n-1} \xrightarrow{d} \cdots \xrightarrow{d} C_0 \xrightarrow{\epsilon} V \longrightarrow 0
\]

in which each \(C_n\) is a G-Line Bundle and each \(d\) is a morphism.

Let \(\pi : G \longrightarrow GL(V)\) be a smooth (aka admissible) representation on a \(k\)-vector space, \(V\). That is, for any compact open modulo the centre subgroup \(K\) of \(G\) the restriction to \(K\) is a direct sum of irreducible finite-dimensional representations, each isomorphism class occurring with finite multiplicity.

Let \(V\) be an admissible representation of \(G\), not necessarily irreducible, but with central character \(\phi\). For \((J, \phi) \in \mathcal{M}_{G, \phi}\) define \(V^{((J, \phi))}\) to be the subspace

\[
V^{((J, \phi))} = \{v \in V \mid j(v) = \phi(j) \cdot v \text{ for all } j \in J\}.
\]

An exact chain complex of continuous \(k[G]\)-modules of the form

\[
\cdots \longrightarrow C_n \xrightarrow{d} C_{n-1} \xrightarrow{d} \cdots \xrightarrow{d} C_0 \xrightarrow{\epsilon} V \longrightarrow 0
\]

is called a monomial resolution of \(V\) if each \(C_n\) is a G-Line Bundle, each \(d\) is a morphism and, in addition, \(\epsilon(C_0^{((J, \phi))}) \subseteq V^{((J, \phi))}\) for each \((J, \phi) \in \mathcal{M}_{G, \phi}\) and, furthermore, the complex of \(k\)-vector spaces

\[
\cdots \longrightarrow C_n^{((J, \phi))} \xrightarrow{d} C_{n-1}^{((J, \phi))} \xrightarrow{d} \cdots \xrightarrow{d} C_0^{((J, \phi))} \xrightarrow{\epsilon} V^{((J, \phi))} \longrightarrow 0
\]

is exact for each \((J, \phi) \in \mathcal{M}_{G, \phi}\).

Our first main result is the following.

**Theorem 2.2.**

Let \(V\) be an admissible representation of \(GL_2 K\) with central character \(\phi\) as in §2.1. Then \(V\) admits a monomial resolution which is unique up to chain homotopy in the category \(k[GL_2 K, \phi]_{\text{mon}}\).
2.3. Some subgroups of $GL_2 K$

Let $K$ be a $p$-adic local field with valuation $v_K : K^* \rightarrow \mathbb{Z}$. We have homomorphisms

$$\det : GL_2 K \rightarrow K^*$$

and

$$v_K \cdot \det : GL_2 K \rightarrow \mathbb{Z}.$$ 

Following ([4] p.75) we may define subgroups of $GL_2 K$ denoted by $SL_2 K, GL_2 K^0$ and $GL_2 K^+$ by

$$SL_2 K = \text{Ker}(\det), \quad GL_2 K^0 = \text{Ker}(v_K \cdot \det), \quad GL_2 K^+ = \text{Ker}(v_K \cdot \det \text{ modulo } 2)$$

so that

$$SL_2 K \subset GL_2 K^0 \subset GL_2 K^+ \subset GL_2 K.$$

As explained in ([4] pp.78/79) and in terms of Tits buildings (i.e. BN-pairs etc) in ([4] p.91) each of the first three groups acts transitively on the vertices of the tree and act on a 1-simplex between adjacent vertices simplicially (i.e. any element sending the 1-simplex to itself does so point-wise).

The following result summarises the properties of the monomial resolutions for representations of $GL_2 K$.

**Theorem 2.4.**

Let $V$ be an admissible representation of $G$ with central character $\phi$ as in §2.1, where $G$ is one of the subgroups of $GL_2 K$ given by $G = SL_2 K, GL_2 K^0$ or $GL_2 K^+$. Then $V$ admits a monomial resolution which is unique up to chain homotopy in the category $k[G,\phi]_{\text{mon}}$.

**Remark 2.5.** In order to describe the construction of the monomial resolution in Theorems 2.2 and 2.4 I need to introduce the functorial bar-monomial resolution for finite-dimensional representations of groups which are finite modulo the centre. This will be introduced in §3 and the construction which proves Theorems 2.2 and 2.4 will be sketched in §4.

The important point to note is that the construction generalises to all $GL_n K$ and when I know more about the Bruhat-Tits building of $GL_n K$ for $n \geq 3$ it is likely that this knowledge will result in a generalisation of Theorem 2.2 to all $GL_n K$, at the moment this is merely a conjecture which appears in §4.

3. The bar-monomial resolution

3.1. For the moment let $G$ be a group which is finite modulo the centre with central character $\phi$. Let $W$ be a $k$-vector space and let $M$ be a left $k[G,\phi]$-Line Bundle. Define another $k[G,\phi]$-Line Bundle on the vector space $W \otimes \bar{M}$ by letting $G$ act only on the $M$-factor, $g(w \otimes m) = w \otimes gm$, and by setting the Lines of $W \otimes M$ to consist of the one-dimensional subspaces $\langle w \otimes L \rangle$ where $w$ is a non-zero vector of $W$, running through a choice of basis for $W$, and $L$ is a Line of $M$. Therefore, if $M’$ is another $k[G,\phi]$-Line Bundle we have a natural isomorphism

$$\text{Hom}_{k[G,\phi]_{\text{mon}}}(W \otimes M, M’) \cong W \otimes \text{Hom}_{k[G,\phi]_{\text{mon}}}(M, M’).$$
providing that $W$ is finite-dimensional.

Let $k[G,\phi]_{\text{mod}}$ denote the category of finite dimensional $k$-modules on which $G$ acts with central character $\phi$. Let $V$ be a representation of $G$ in $k[G,\phi]_{\text{mod}}$ and let $S$ be a Line Bundle given by a direct sum of some $\text{Ind}_H^G(\phi)$’s with $(H, \phi) \in \mathcal{M}_{G,\phi}$.

If $V$ is the forgetful functor from Line Bundles to $G$-modules set

$$W_i = \text{Hom}_{k[G,\phi]_{\text{mod}}}(V(S), V) \otimes \text{Hom}_{k[G,\phi]_{\text{mon}}}(S, S)^{\otimes i},$$

assuming that $W_i$ is finite-dimensional.

We have left $k[G,\phi]$-monomial morphisms, defined by the obvious formulae,

$$d_0, d_1, \ldots, d_i : W_i \otimes S \longrightarrow W_{i-1} \otimes S$$

for $i \geq 1$ and a left $k[G,\phi]$-module homomorphism

$$\epsilon : \text{Hom}_{k[G,\phi]_{\text{mod}}}(V(S), V) \otimes S \longrightarrow V$$

given by $\epsilon(f \otimes s) = f(s)$. Let $d$ be given by the alternating sum

$$d = \sum_{j=0}^i (-1)^j d_j$$

and choice $S$ to be the direct sum of one copy of each $\text{Ind}_H^G(\phi)$ for $(H, \phi) \in \mathcal{M}_{G,\phi}$.

**Theorem 3.2.**

In the notation of §3.1 the complex

$$\ldots \longrightarrow W_i(S) \otimes S \overset{d}{\longrightarrow} W_{i-1}(S) \otimes S \longrightarrow \ldots \longrightarrow W_0 \otimes S \overset{\epsilon}{\longrightarrow} V \longrightarrow 0$$

is a canonical left $k[G,\phi]$-monomial resolution of $V$, which is natural with respect to homomorphisms of groups $G$.

4. **Constructing admissible representations via monomial resolutions**

4.1. Let $G$ be a locally $p$-adic Lie group such as $G = GL_nK$ for $K$ a local field. Let $Y$ be a simplicial complex upon which $G$ acts simplicially and in which the stabiliser $H_\sigma = \text{stab}_G(\sigma)$ is compact, open modulo the centre of $G$. An example of this is $GL_nK$ acting on a suitable subdivision of its building. Let $V$ be an irreducible, admissible representation of $G$ over $k$. For each simplex $\sigma$ we have a $kH_\sigma$-bar-monomial resolution

$$W_{s, H_\sigma} \longrightarrow V \longrightarrow 0,$$

obtained as the direct sum of bar-monomial resolutions of the finite-dimensional irreducible summands of $V$ restricted to $H_\sigma$. Form the graded $k$-vectorspace which in degree $m$ is equal to

$$M_m = \bigoplus_{\alpha + n = m} W_{\alpha, H_\sigma^n}.$$ 

If $\sigma^{n-1}$ is a face of $\sigma^n$ there is an inclusion $H_{\sigma^n} \subseteq H_{\sigma^{n-1}}$. Therefore there is a monomial chain map

$$i_{H_{\sigma^n}, H_{\sigma^{n-1}}} : W_{s, H_{\sigma^n}} \longrightarrow W_{s, H_{\sigma^{n-1}}}$$
such that
\[ i_{H_{n-1}, H_{n-2}} \circ i_{H_n, H_{n-1}} = i_{H_n, H_{n-2}}. \]
If \( \sigma^{n-1} \) is a face of \( \sigma^n \) let \( d(\sigma^{n-1}, \sigma^n) \) denote the incidence degree of \( \sigma^{n-1} \) in \( \sigma^n \); this is \( \pm 1 \). In the simplicial chain complex of \( Y \)
\[ d(\sigma^n) = \sum_{\sigma^{n-1} \text{ face of } \sigma^n} d(\sigma^{n-1}, \sigma^n) \sigma^{n-1}. \]
For \( x \in W_{\alpha, H_\sigma} \) write
\[ d_Y(x) = \sum_{\sigma^{n-1} \text{ face of } \sigma^n} d(\sigma^{n-1}, \sigma^n) i_{H_{\sigma}, H_{\sigma^{n-1}}}(x). \]
Let \( d_{\sigma^n} : W_{\alpha, H_\sigma} \to W_{\alpha-1, H_\sigma} \) denote the differential in the monomial resolution.
Define \( d : M_n \to M_{m-1} \) when \( m = \alpha + n \) by
\[ d(x) = d_Y(x) + (-1)^n d_{\sigma^n}(x). \]
Therefore we have
\[ d(d(x)) = d(\sum_{\sigma^{n-1} \text{ face of } \sigma^n} d(\sigma^{n-1}, \sigma^n) i_{H_{\sigma}, H_{\sigma^{n-1}}}(x)) + d((-1)^n d_{\sigma^n}(x)) \]
\[ = \sum_{\sigma^{n-1} \text{ face of } \sigma^n} d(\sigma^{n-1}, \sigma^n) i_{H_{\sigma}, H_{\sigma^{n-1}}}(x) \]
\[ + \sum_{\sigma^{n-1} \text{ face of } \sigma^n} d(\sigma^{n-1}, \sigma^n) i_{H_{\sigma}, H_{\sigma^{n-1}}}(x) \]
\[ + (-1)^n \sum_{\sigma^{n-1} \text{ face of } \sigma^n} d(\sigma^{n-1}, \sigma^n) d_{\sigma^{n-1}}(i_{H_{\sigma}, H_{\sigma^{n-1}}}(x)) \]
\[ + (-1)^n d_{\sigma^n}((-1)^n d_{\sigma^n}(x)) \]
\[ = \sum_{\sigma^{n-1} \text{ face of } \sigma^n} d(\sigma^{n-1}, \sigma^n) i_{H_{\sigma}, H_{\sigma^{n-1}}}(x) \]
\[ + (-1)^n \sum_{\sigma^{n-1} \text{ face of } \sigma^n} d(\sigma^{n-1}, \sigma^n) i_{H_{\sigma}, H_{\sigma^{n-1}}}(d_{\sigma^n}(x)) \]
\[ + (-1)^n \sum_{\sigma^{n-1} \text{ face of } \sigma^n} d(\sigma^{n-1}, \sigma^n) i_{H_{\sigma}, H_{\sigma^{n-1}}}(d_{\sigma^n}(x)) \]
\[ + d_{\sigma^n}(d_{\sigma^n}(x)) \]
\[ = \sum_{\sigma^{n-1} \text{ face of } \sigma^n} d(\sigma^{n-1}, \sigma^n) i_{H_{\sigma}, H_{\sigma^{n-1}}}(x) \]
\[ = 0 \]
because, as is well-known, for each pair \((\sigma^n, \sigma^{n-2})\) the sum
\[
\sum_{\sigma^{n-2} \text{ face of } \sigma^{n-1}} d(\sigma^{n-2}, \sigma^{n-1}) \cdot d(\sigma^{n-1}, \sigma^n) = 0.
\]

**Theorem 4.2.** Existence of monomial resolutions for \(GL_2 k\)

\((M_+, d)\) is a \(kG\)-monomial complex when \(G = GL_n k\) and \(Y\) is the building of \(G\).

When \(G = GL_2 k\) \((M_+, d)\) is a monomial resolution of the admissible representation \(V\) and, as such, is unique up to chain homotopy equivalence.

**Conjecture 4.3.** Existence of monomial resolutions for \(GL_n k\)

For \(n \geq 3\), \(k\) local and \(G = GL_n k\)

\[
\rightarrow M_i \xrightarrow{d} M_{i-1} \xrightarrow{d} \ldots \xrightarrow{d} M_0 \xrightarrow{\epsilon} V \rightarrow 0
\]
is a \(kG\)-monomial resolution. That is, for each \((H, \phi) \in M_G\)

\[
\rightarrow M_i^{((H, \phi))} \xrightarrow{d} M_{i-1}^{((H, \phi))} \xrightarrow{d} \ldots \xrightarrow{d} M_0^{((H, \phi))} \xrightarrow{\epsilon} V^{((H, \phi))} \rightarrow 0
\]
is an exact sequence of \(k\)-vector spaces.

**Remark 4.4.** From the monomial resolution one recovers \(V\) as the zero-th homology.

5. RELATION OF THE MONOMIAL RESOLUTION WITH \(\pi_K\)-ADIC LEVELS

5.1. In this section suppose that \(V\) is defined on a vector space over \(k\), an algebraically closed field of characteristic zero.

For \(n \geq 1\) consider the compact open modulo the centre subgroup \(J_n = K^* \cdot U_n\) where \(U_n = 1 + \pi^n k M_2 O_k\). In this section we shall assume that \(n\) is large enough such that the restriction of the central character \(\phi\) to \(K^* \cap U_n\) is trivial. In this case we shall denote by

\[
\phi : K^* \cdot U_n \longrightarrow k^*
\]
the character which is given by \(\overline{\phi}\) on \(K^*\) is trivial on \(U_n\).

**Theorem 5.2.**

In the situation of §5.1

\[
M_+^{((K^* \cdot U_n, \phi))} \longrightarrow V^{((K^* \cdot U_n, \phi))}
\]
is a \(K^* \cdot U_n / U_n\)-monomial resolution, whose chain homotopy class contains a finitely generated \(K^* \cdot U_n / U_n\)-monomial resolution of finite length.

**Proof**

This follows from the fact that \(M_+ \longrightarrow V \longrightarrow 0\) is a monomial resolution of \(V\). \(\square\)

**Remark 5.3.** If Conjecture 4.3 were true then the analogue of Theorem 4.2 for \(GL_n k\) would be true for all \(n \geq 1\).
6. Epsilon factors and L-functions

6.1. If \( V \) is an admissible representation of \( GL_2 K \) and \( M \longrightarrow V \) is a monomial resolution as in Theorem 4.2 one may construct epsilon factors for \( V \) by applying an integral to each Line given by an integral for character values which in the finite finite case specialises to the Kondo Gauss sums. These integrals respect induction from one compact, open modulo the centre subgroup to another.

I am assuming an analogue of the result concerning wild epsilon factors modulo \( p \)-power roots of unity \([3]\) holds for all but a finite set of Lines with the result that a well-defined epsilon factor modulo \( p \)-power roots of unity is defined by a finite product of Kondo-style Gauss sums. Here I ought to mention that I slightly disagree with a fundamental result in \([3]\) (see \([5]\)) so the epsilon factor I propose may only be well defined up to \( \pm 1 \) times a \( p \)-power root of unity.

I have yet to develop the approach of Tate’s thesis to each Line to get the L-functions.

These methods would apply to \( GL_n K \) if Conjecture 4.3 holds.

7. Galois descent for \( GL_2 K \)

7.1. Suppose that \( K \) is a \( p \)-adic local field and \( \rho : GL_2 K \longrightarrow GL(V) \) is a complex, irreducible admissible representation and that \( k \) is the complex field. Let \( K/F \) be a Galois extension and suppose that \( z^* \rho \) is equivalent to \( \rho \) for each \( z \in \text{Gal}(K/F) \). Therefore for \( z \in \text{Gal}(K/F) \) there exists \( X_z \in GL(V) \) such that

\[
X_z \rho(g) X_z^{-1} = \rho(z(g))
\]

for all \( g \in GL_2 K \). Therefore if \( z, z_1 \in \text{Gal}(K/F) \) replacing \( g \) by \( z_1(g) \) gives

\[
X_z \rho(z_1(g)) X_z^{-1} = \rho(z z_1(g))
\]

and so

\[
X_z \rho(z_1(g)) X_z^{-1} = X_z X_{z_1} \rho(g) X_{z_1}^{-1} X_z^{-1} = X_{z z_1} \rho(g) X_{z z_1}^{-1}.
\]

By Schur’s Lemma \( X_{z_1}^{-1} X_z^{-1} X_{z z_1} \) is a scalar matrix and so

\[
f(z, z_1) = X_{z_1}^{-1} X_z^{-1} X_{z z_1}
\]

is a function from \( \text{Gal}(K/F) \times \text{Gal}(K/F) \) to \( \mathbb{C}^* \). In fact, \( f \) is a 2-cocycle.

By a result of Tate \( H^2(F; \mathbb{C}^*) = 0 \) if \( K \) is local or global. Therefore there exists a finite Galois extension \( E/F \) such that the 2-cocycle induced by \( f \)

\[
f' : \text{Gal}(E/F) \times \text{Gal}(E/F) \longrightarrow \mathbb{C}^*
\]

is a coboundary \( f' = dF \). Then \( z \mapsto X_z F(z) \) is a homomorphism from \( \text{Gal}(E/F) \longrightarrow GL(V) \).

Recall that the semi-direct product \( \text{Gal}(E/F) \times GL_2 K = G \) is given by the set \( \text{Gal}(E/F) \times GL_2 K \) with the product defined by

\[
(h_1, g_1) \cdot (h_2, g_2) = (h_1 h_2, g_1 h_1(g_2)).
\]
The map
\[ \tilde{\rho} : \text{Gal}(E/F) \times GL_n K \to GL(V) \]
which sends \((z, g)\) to \(\rho(g)X_zF(z)\) is an irreducible admissible representation extending \(\rho\). Any two such extensions differ by twisting via a homomorphism \(\text{Gal}(E/F) \to C^*\) for some \(E\).

The action of \(\text{Gal}(K/F)\) on \(K \oplus K\) preserves the lattice \(L = O_K \oplus O_K\) and \(L' = O_K \oplus \pi_K O_K\) and their stabilisers under the tree-action \(H_1\) and \(H_2\). Therefore the Galois action fixes the canonical fundamental domain on the tree and the semi-direct product acts on the tree of \(GL_2 K\), extending the action of \(GL_2 K\).

Replacing \(H_1\) and \(H_2\) by \(\text{Gal}(E/K) \times GL_n F\) yields the following result.

**Theorem 7.2.**
There exists a monomial resolution of \(\tilde{\rho}\) which is unique up to chain homotopy and satisfies the analogue of Theorem 4.2.

**7.3. The Galois descent yoga**

Take \(\rho\) and form the monomial resolution of \(\tilde{\rho}\) as in Theorem 7.2. Quotient out the monomial complex by the Lines whose stabiliser group is not sub-conjugate in the semi-direct product to \(\text{Gal}(E/F) \times GL_n F\). This is a monomial complex for the semi-direct product which originates, via induction, with \(\text{Gal}(E/F) \times GL_n F\).

In ONE case of finite general linear groups this yoga is equivalent to Shintani descent. See [6].

I conjecture (on the basis of only one case!!) that Galois base change for admissible representations of \(GL_n\) of local fields can be described in terms of the above yoga with monomial resolutions.

**8. Restricted tensor products and global automorphic representations**

**8.1.** Automorphic representations of \(GL_2 F\) where \(F\) is a number field are constructed by the tensor product theorem. For convenience let \(F = \mathbb{Q}\), the rationals, so that we can just refer to [2]. Here is the tensor product theorem in that case - the reader is referred to [2] for details. Suffice to say that the key to the construction is the fact that \(V^{(GL_2 \mathbb{Z}_p, 1)}\) is one-dimensional for almost all primes \(p\).


Let \((\pi, V)\) denote an irreducible admissible \((U(gl_2 \mathbb{C}), K_\infty) \times GL_2 A_{fin}\)-module. Let \(\{q_1, \ldots, q_m\}\) be the finite set of primes where \(\pi\) is ramified. Let \(S = \{\infty, q_1, \ldots, q_m\}\). Then there exists

(i) an irreducible admissible \((U(gl_2 \mathbb{C}), K_\infty)\)-module \((\pi_\infty, V_\infty)\),
(ii) an irreducible admissible representation \((\pi_p, V_p)\) of \(GL_2 \mathbb{Q}_p\) for each finite prime \(p\),
(iii) a non-zero vector $v_p^0 \in V_p^{GL_2\mathbb{Q}_p}$ for each prime $\not\in S$ such that

$$\pi \cong \bigotimes_{v \leq \infty} \pi_v.$$  


8.3. Restricted tensor products of monomial resolutions

The restricted tensor product construction of Theorem 8.1 makes sense when applied to monomial resolutions of local admissible irreducible representation of $GL_2\mathbb{Q}_p$. One just restricts the tensor product of the monomial resolutions to lie in $M_{\pi}((\mathbb{Q}_p, GL_2\mathbb{Z}_p, \varphi))$ for primes for which $V((GL_2\mathbb{Z}_p, 1))$ is one-dimensional.

This constructs a global monomial resolution, unique up to chain homotopy, for each global automorphic representation of $GL_2\mathbb{Q}$ or, more generally, for $GL_2F$ for any number field, $F$. Handling the Archimedean places in the usual manner (see [2]) gives a construction of automorphic representations of $GL_2F$ via monomial resolutions.

REFERENCES


